ABOUT THE LOGARITHM FUNCTION OVER THE MATRICES

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ABSTRACT. We prove the following results: let x, y be (n, n) complex matrices such that x, y, xy have no eigenvalue in $]-\infty, 0]$ and $log(xy) = log(x) + log(y)$. If $n = 2$, or if $n \geq 3$ and x, y are simultaneously triangularizable, then x, y commute. In both cases we reduce the problem to a result in complex analysis.

1. Introduction

$\mathbb{Z}^*$ refers to the non-zero integers.
Let u be a complex number. Then $Re(u), Im(u)$ refer to the real and imaginary parts of u; if $u \notin]-\infty, 0]$ then $arg(u) \in]-\pi, \pi[$ refers to its principal argument.

1.1. **Basic facts about the logarithm.** Let x be a complex (n, n) matrix which hasn't any eigenvalue in $]-\infty, 0]$. Then $log(x)$, the x-principal logarithm, is the (n, n) matrix a such that:
$e^a = x$ and the eigenvalues of a lie in the strip $\{z \in \mathbb{C} : Im(z) \in]-\pi, \pi[\}$.
$log(x)$ always exists and is unique; moreover $log(x)$ may be written as a polynomial in x.

Now we consider two matrices x, y which have no eigenvalue in $]-\infty, 0]$:
- If x, y commute then x, y are simultaneously triangularizable and we may associate pairwise their eigenvalues $(\lambda_j), (\mu_j)$; if moreover $\forall j, |arg(\lambda_j) + arg(\mu_j)| < \pi$, then $log(xy) = log(x) + log(y)$.
- Conversely if xy has no eigenvalue in $]-\infty, 0]$ and $log(xy) = log(x) + log(y)$ then do x, y commute ? We will prove that it's true for $n = 2$ (theorem 1) or, for all n, if x, y are simultaneously triangularizable (theorem 2). But if $n > 2$, then we don't know the answer in the general case.

1.2. **Lemma 1.** Let x, y be two complex (n, n) matrices such that x, y haven't any eigenvalue in $]-\infty, 0]$ and $log(x)log(y) = log(y)log(x)$.
Then x, y commute.

Proof. The principal logarithm over $\mathbb{C} \setminus]-\infty, 0]$ is one to one; thus, using Hermite's interpolation formula, x or y may be written as a polynomial in $log(x)$ or $log(y)$. $\square$

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2. Dimension 2

2.1. **Principle of the proof.** The proof is based on the two next propositions. The first one is a corollary of a Morinaga and Nono's result ([1, p. 356]); the second is a technical result using complex analysis.

Proposition 1. Let $\mathcal{U} = \{u \in \mathbb{C}^* : e^u = 1 + u\}$.
Let a, b be two $(2,2)$ complex matrices such that $e^{a+b} = e^a e^b$ and $ab \neq ba$; let $spectrum(a) = \{\lambda_1, \lambda_2\}, spectrum(b) = \{\mu_1, \mu_2\}$.
Then one of the three following *item* is fulfilled:
(1) $\lambda_1 - \lambda_2 \in 2i\pi\mathbb{Z}^*$ and $\mu_1 - \mu_2 \in 2i\pi\mathbb{Z}^*$.
(2) One of the following complex numbers $\pm(\lambda_1 - \lambda_2)$, $\pm(\mu_1 - \mu_2)$ is in $\mathcal{U}$.
(3) a and b are simultaneously similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda + u \end{pmatrix}$ and $\begin{pmatrix} \mu + v & 1 \\ 0 & \mu \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C}$, $u, v \in \mathbb{C}^*, u \neq v$ and $\dfrac{e^u - 1}{u} = \dfrac{e^v - 1}{v} \neq 0$.

Proposition 2. Let u, v be two distinct, non zero complex numbers such that $\dfrac{e^u - 1}{u} = \dfrac{e^v - 1}{v} \neq 0$, $|Im(u)| < 2\pi, |Im(v)| < 2\pi$.
Then necessarily $|Im(u) - Im(v)| \geq 2\pi$.

Proof. Assume that we can choose these u, v such that $|Im(u) - Im(v)| < 2\pi$. Let $\lambda = \dfrac{e^u - 1}{u}$ and let f be the holomorphic function: $f(z) = e^z - \lambda z - 1$.

Now we show that there exists $a \in]0, 2\pi[$ such that $Im(u), Im(v)$ are in $]-a, 2\pi - a[$ and f hasn't any zero with imaginary part $-a$ or $2\pi - a$.
If it's false then f admits an infinity of zeros in the strip $\{z : Im(z) \in]-2\pi, 2\pi[\}$:
Case 1: we can extract a sequence of zeros z_k such that $Re(z_k) \to -\infty$; then $f(z_k) \sim -\lambda z_k$, a contradiction.
Case 2: we can extract a sequence of zeros z_k such that $Re(z_k) \to +\infty$; then $f(z_k) \sim e^{z_k}$, a contradiction.

Let r be a big positive real such that $Re(u), Re(v)$ are in $]-r, r[$ and $\triangle$ be the rectangle $\{z : -r \leq Re(z) \leq r, -a \leq Im(z) \leq 2\pi - a\}$. The oriented edge $\partial\triangle$ consists of four parts: $h_1 = \{x + i(2\pi - a) : x \text{ from } r \text{ to } -r\}$, $v_1 = \{-r + iy : y \text{ from } 2\pi - a \text{ to } -a\}$, $h_2 = \{x - ia : x \text{ from } -r \text{ to } r\}$, $v_2 = \{r + iy : y \text{ from } -a \text{ to } 2\pi - a\}$. f admits in $\triangle$ at least three zeros: $0, u, v$. Thus $\dfrac{1}{2i\pi} \int_{\partial\triangle} \dfrac{f'(z)}{f(z)} dz = I(f(\partial\triangle), 0) \geq 3$ where I refers to the index function.

$f(z + 2i\pi) - f(z) = -\lambda 2i\pi$; then $\{f(h_1), f(h_2)\}$ is in tubular form; moreover $f(h_1)$ and $f(h_2)$ are isometric to a parametric curve in the form $\{(e^t - \sigma t, \tau t) : t \in [-r, r]\}$ where σ, τ are real; we can choose a such that $\sigma, \tau \in \mathbb{R}^*$; thus for $j \in \{1, 2\}$ $|\dfrac{1}{2i\pi} \int_{h_j} \dfrac{f'(z)}{f(z)} dz| \leq 1 - \rho$ with $\rho = \dfrac{1}{2\pi} arctan(|\dfrac{\tau}{\sigma}|)$.

We can choose r, a such that $f'(z) \neq 0$ on $\partial\triangle$; thus $f(h_1), f(h_2)$ intersect perpendicularly $f(v_1)$ and $f(v_2)$. If $z \in v_1$ and $r \to +\infty$ then $f(z) = -\lambda z - 1 + O(e^{-r})$; $f(v_1)$ is close to a segment of fixed direction and length. If $z \in v_2$ and $r \to +\infty$ then $f(z) = e^z + O(r)$; $f(v_2)$ is close to an anticlockwise circle of radius e^r containing 0.

Therefore $\frac{1}{2i\pi}(\int_{h_1}\frac{f'(z)}{f(z)}dz+\int_{h_2}\frac{f'(z)}{f(z)}dz)\leq 2-2\rho$, $\frac{1}{2i\pi}(\int_{v_1}\frac{f'(z)}{f(z)}dz)\approx 0$, $\frac{1}{2i\pi}(\int_{v_2}\frac{f'(z)}{f(z)}dz)\approx 1$; what is contradictory with $I(f(\partial\triangle),0)\geq 3$. $\square$

2.2. **Theorem 1.** Let x,y be two $(2,2)$ complex matrices such that x,y,xy haven't any eigenvalue in $]-\infty,0]$ and $log(xy)=log(x)+log(y)$. Then x,y commute.

Proof. We assume that $xy\neq yx$. $e^{log(x)}e^{log(y)}=e^{log(x)+log(y)}$; using lemma 1, $log(x)log(y)\neq log(y)log(x)$; thus we may use Proposition 1; it's wellknown that $u\in\mathcal{U}$ implies that $|Im(u)|>2\pi$; then, according to the logarithm definition, $a=log(x)$ and $b=log(y)$ satisfy *item* (3). Moreover the conditions $|Im(u)|<2\pi, |Im(v)|<2\pi, |Im(u)-Im(v)|<2\pi$ are necessarily fulfilled. Proposition 2 proves that these conditions can't be all satisfied. $\square$

3. Dimension n

I refers to the identity matrix of dimension $n-1$. Let ϕ be the holomorphic function: $\phi: z\rightarrow\frac{e^z-1}{z}, \phi(0)=1$.

Remark 1. We have shown in part 2 that if u,v are complex numbers such that $|Im(u)|<2\pi, |Im(v)|<2\pi, |Im(u-v)|<2\pi$ and $\phi(u)=\phi(v)$, then $u=v$.

We'll use the following to prove our second main result.

3.1. **Proposition 3.** Let $a=\begin{pmatrix}a_0 & u\\ 0 & \alpha\end{pmatrix}, b=\begin{pmatrix}b_0 & v\\ 0 & \beta\end{pmatrix}$ be two complex (n,n) matrices where α,β are complex numbers and a_0,b_0 are $(n-1,n-1)$ complex matrices which commute; let $spectrum(a_0-\alpha I)=(\alpha_i)_{i\leq n-1}$, $spectrum(b_0-\beta I)=(\beta_i)_{i\leq n-1}$. If $e^{a+b}=e^ae^b$ and $ab\neq ba$ then one of the following *item* must be satisfied:
(4) $\exists i:\beta_i\neq 0$ and $\phi(\alpha_i+\beta_i)=\phi(\alpha_i)$.
(5) $\exists i:\alpha_i\neq 0,\beta_i=0$ and $\phi(-\alpha_i)=1$.

Proof. We may assume that a_0,b_0 are upper triangular. Let $a_1=a_0-\alpha I, b_1=b_0-\beta I, w=a_1v-b_1u=[w_1,\cdots,w_{n-1}]^T$. Thus $ab\neq ba$ iff $w\neq 0$.
Then $e^a=e^\alpha\begin{pmatrix}e^{a_1} & \phi(a_1)u\\ 0 & 1\end{pmatrix}, e^b=e^\beta\begin{pmatrix}e^{b_1} & \phi(b_1)v\\ 0 & 1\end{pmatrix}$,
$e^{a+b}=e^{\alpha+\beta}\begin{pmatrix}e^{a_1+b_1} & \phi(a_1+b_1)(u+v)\\ 0 & 1\end{pmatrix}$; therefore $e^{a+b}=e^ae^b$ iff
(6) $(\phi(a_1+b_1)-\phi(a_1))u=(e^{a_1}\phi(b_1)-\phi(a_1+b_1))v$.
$(e^{a_1}\phi(b_1)-\phi(a_1+b_1))b_1=(\phi(a_1+b_1)-\phi(a_1))a_1$ and (6) imply that $(e^{a_1}\phi(b_1)-\phi(a_1+b_1))b_1v=(\phi(a_1+b_1)-\phi(a_1))a_1v=(\phi(a_1+b_1)-\phi(a_1))b_1u$; thus
(7) $(\phi(a_1+b_1)-\phi(a_1))w=0$.
We have also $e^b=e^{-a}e^{a+b}$; then we can prove by the same method that
(8) $(\phi(b_1)-\phi(-a_1))w=0$.
There exists k such that $w_k\neq 0$ and if $j>k$ then $w_j=0$. Therefore (7),(8) imply that $\phi(\alpha_k+\beta_k)=\phi(\alpha_k)$ and $\phi(\beta_k)=\phi(-\alpha_k)$; we are done except if $\alpha_k=\beta_k=0$.

Now we assume that $\alpha_k=\beta_k=0$. $\phi(a_1+b_1)-\phi(a_1)=\frac{1}{2}b_1(I+P(a_1,b_1)), e^{a_1}\phi(b_1)-\phi(a_1+b_1)=\frac{1}{2}a_1(I+P(a_1,b_1))$ where P is an analytic function, defined on $\mathbb{C}^2$,

which satisfies $P(0,0)=0$. (6) can be rewritten as $(I+P(a_1,b_1))w=0$. Therefore $(1+P(0,0))w_k=0$, a contradiction. $\square$

3.2. **Theorem 2.** Let x,y be (n,n) complex matrices such that x,y,xy haven't any eigenvalue in $]-\infty,0]$ and $log(xy)=log(x)+log(y)$. If moreover x,y are simultaneously triangularizable then $xy=yx$.

Proof. We assume that x,y are upper-triangular and $xy\neq yx$; we prove inductively the result for $n\geq 2$. $x=\begin{pmatrix} x_0 & ? \\ 0 & \lambda \end{pmatrix}$, $y=\begin{pmatrix} y_0 & ? \\ 0 & \mu \end{pmatrix}$ where x_0,y_0 are $(n-1,n-1)$ upper triangular matrices which haven't any eigenvalue in $]-\infty,0]$ and $\lambda,\mu\in\mathbb{C}\backslash]-\infty,0]$. The matrices $a=log(x),b=log(y)$ are polynomials in x or y, thus they are upper-triangular in form $a=\begin{pmatrix} log(x_0) & ? \\ 0 & log(\lambda) \end{pmatrix}$, $b=\begin{pmatrix} log(y_0) & ? \\ 0 & log(\mu) \end{pmatrix}$. Thus $log(x_0y_0)=log(x_0)+log(y_0)$; according to the recurrence hypothesis $x_0y_0=y_0x_0$ and then $log(x_0)log(y_0)=log(y_0)log(x_0)$. Moreover $e^{a+b}=e^ae^b$ and, from lemma 1, $ab\neq ba$.

Now we use Proposition 3 with $\alpha=log(\lambda),\beta=log(\mu),a_0=log(x_0),b_0=log(y_0)$. Here $\alpha_i,\beta_i,\alpha_i+\beta_i$ have imaginary parts in $]-2\pi,2\pi[$ and according to Remark 1, *item* (4),(5) can't be satisfied. $\square$

We conclude with an easy result.

3.3. **Proposition 4.** Let x,y be two positive definite hermitian (n,n) matrices so that $log(xy)=log(x)+log(y)$. Then $xy=yx$.

Proof. $log(xy)$ exists because $spectrum(xy)\subset]0,\infty[$; $a=log(x),b=log(y)$ are hermitian matrices such that $e^{a+b}=e^ae^b$. Moreover $e^{a+b}=(e^{a+b})^*=e^be^a$ and $e^ae^b=e^be^a$ or $xy=yx$. $\square$

Remark. It's wellknown that if a,b are bounded self adjoint operators on a complex Hilbert space, then $e^{a+b}=e^ae^b$ implies that $ab=ba$. (*cf.* [2, Corollary 1]).

4. Conclusion

When $n=2$, we know how to characterize the complex (n,n) matrices a,b such that $ab\neq ba$ and $e^{a+b}=e^ae^b$; it allowed us to bring back our problem to a result of complex analysis. Unfortunately, if $n\geq 3$, the classification of such matrices is unknown. For this reason we can't prove, in this last case, the hoped result without supplementary assumption.

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